

THE HIGHER DUALS OF CERTAIN CLASS OF BANACH ALGEBRAS

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ABSTRACT. Given a Banach space A and fix a non-zero $\varphi \in A^*$ with $\|\varphi\| \leq 1$. Then the product $a \cdot b = \langle \varphi, a \rangle b$ turning A into a Banach algebra which will be denoted by ${}_{\varphi}A$. Some of the main properties of ${}_{\varphi}A$ such as Arens regularity, n -weak amenability and semi-simplicity are investigated.

1. INTRODUCTION

This paper had its genesis in a simple beautiful example of Zhang [6, Page 507]. For an infinite set S he equipped $l^1(S)$ with the algebra product $a \cdot b = a(s_0)b$ ($a, b \in l^1(S)$), where s_0 is a fixed element of S . He used this as a Banach algebra which is $(2n-1)$ -weakly amenable but is not $(2n)$ -weakly amenable for any $n \in \mathbb{N}$. Here we study a more general form of this example. Indeed, we equip a non-trivial product on a general Banach space turning it to a Banach algebra whose study has significance in many respects. For instance, it can serve as a source of (counter-)examples for various purposes in functional analysis.

Given a Banach space A and fix a non-zero $\varphi \in A^*$ with $\|\varphi\| \leq 1$. Then the product $a \cdot b = \langle \varphi, a \rangle b$ turning A into a Banach algebra which will be denoted by ${}_{\varphi}A$. Trivially ${}_{\varphi}A$ has a left identity (indeed, every $e \in {}_{\varphi}A$ with $\langle \varphi, e \rangle = 1$ is a left identity), while it has no bounded approximate identity in the case where $\dim(A) \geq 2$. Now the Zhang's example can be interpreted as an special case of ours. Indeed, he studied ${}_{\varphi_{s_0}}l^1(S)$, where $\varphi_{s_0} \in l^\infty(S)$ is the characteristic at s_0 . Here, among other things, we focus on the higher duals of ${}_{\varphi}A$ and investigate various notions of amenability for ${}_{\varphi}A$. In particular, we prove that for every $n \in \mathbb{N}$, ${}_{\varphi}A$ is $(2n-1)$ -weakly amenable but it is not $(2n)$ -weakly amenable for any n , in the case where $\dim(\ker \varphi) \geq 2$.

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2. THE RESULTS

Before we proceed for the main results we need a bit preliminaries. As we shall be concerned to the Arens products \square and \diamond on the bidual A^{**} of a Banach algebra A , let us to introduce these products in the following steps.

Let $a, b \in A, f \in A^*$ and $m, n \in A^{**}$.

$$\begin{aligned} \langle f \cdot a, b \rangle &= \langle f, ab \rangle & \langle b, a \cdot f \rangle &= \langle ba, f \rangle \\ \langle n \cdot f, a \rangle &= \langle n, f \cdot a \rangle & \langle a, f \cdot n \rangle &= \langle a \cdot f, n \rangle \\ \langle m \square n, f \rangle &= \langle m, n \cdot f \rangle & \langle f, m \diamond n \rangle &= \langle f \cdot m, n \rangle. \end{aligned}$$

If \square and \diamond coincide on the whole of A^{**} then A is called Arens regular. For the brevity of notation we use the same symbol “ \cdot ” for the various module operations linking A , such as A^* , A^{**} and also as well for the n^{th} dual $A^{(n)}$, ($n \in \mathbb{N}$). The main properties of these products and various A -module operations are detailed in [1]; see also [5].

A derivation from a Banach algebra A to a Banach A -module X is a bounded linear mapping $D : A \rightarrow X$ such that $D(ab) = D(a)b + aD(b)$ ($a, b \in A$). For each $x \in X$ the mapping $\delta_x : a \rightarrow ax - xa$, ($a \in A$) is a bounded derivation, called an inner derivation. The concept of n -weak amenability was introduced and intensively studied by Dales *et al.* [2]. A Banach algebra \mathcal{A} is said to be n -weakly amenable ($n \in \mathbb{N}$) if every derivation from \mathcal{A} into $\mathcal{A}^{(n)}$ is inner. Trivially, 1-weak amenability is nothing else than weak amenability. A derivation $D : A \rightarrow A^*$ is called cyclic if $\langle D(a), b \rangle + \langle D(b), a \rangle = 0$ ($a, b \in A$). If every bounded cyclic derivation from A to A^* is inner then A is called cyclicly amenable which was studied by Grønbaek [4]. Throughout the paper we usually identify an element of a space with its canonical image in its second dual.

Now we come to ${}_{\varphi}A$. A direct verification reveals that for $a \in A, f \in ({}_{\varphi}A)^*$ and $m, n \in ({}_{\varphi}A)^{**}$,

$$\begin{aligned} f \cdot a &= \langle \varphi, a \rangle f & a \cdot f &= \langle f, a \rangle \varphi \\ n \cdot f &= \langle n, f \rangle \varphi & f \cdot n &= \langle n, \varphi \rangle f \\ m \square n &= \langle m, \varphi \rangle n & m \diamond n &= \langle m, \varphi \rangle n. \end{aligned}$$

The same calculation gives the ${}_{\varphi}A$ -module operations of $({}_{\varphi}A)^{(2n-1)}$ and $({}_{\varphi}A)^{(2n)}$ as follows,

$$\begin{aligned} F \cdot a &= \langle \varphi, a \rangle F & a \cdot F &= \langle F, a \rangle \varphi & (F \in ({}_{\varphi}A)^{(2n-1)}) \\ G \cdot a &= \langle G, \varphi \rangle a & a \cdot G &= \langle \varphi, a \rangle G & (G \in ({}_{\varphi}A)^{(2n)}). \end{aligned}$$

We commence with the next straightforward result, most parts of which are based on the latter observations on the various duals of ${}_{\varphi}A$.

Proposition 1. (i) ${}_{\varphi}A$ is Arens regular and $({}_{\varphi}A)^{**} = {}_{\varphi}(A^{**})$. Furthermore, for each $n \in \mathbb{N}$, $({}_{\varphi}A)^{(2n)}$ is Arens regular.

(ii) $({}_{\varphi}A)^{**} \cdot {}_{\varphi}A = {}_{\varphi}A$ and ${}_{\varphi}A \cdot ({}_{\varphi}A)^{**} = ({}_{\varphi}A)^{**}$; in particular, ${}_{\varphi}A$ is a left ideal of $({}_{\varphi}A)^{**}$.

(iii) $({}_{\varphi}A)^* \cdot {}_{\varphi}A = ({}_{\varphi}A)^*$ and ${}_{\varphi}A \cdot ({}_{\varphi}A)^* = \mathbb{C}\varphi$.

As ${}_{\varphi}A$ has no approximate identity, in general, it is not amenable. The next result investigates n -weak amenability of ${}_{\varphi}A$.

Theorem 2. For each $n \in \mathbb{N}$, ${}_{\varphi}A$ is $(2n - 1)$ -weakly amenable, while in the case where $\dim(\ker \varphi) \geq 2$, ${}_{\varphi}A$ is not $(2n)$ -weakly amenable for any $n \in \mathbb{N}$.

Proof. Let $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^{(2n-1)}$ be a derivation and let $a, b \in {}_{\varphi}A$. Then

$$\langle \varphi, a \rangle D(b) = D(ab) = D(a)b + aD(b) = \langle \varphi, b \rangle D(a) + \langle D(b), a \rangle \varphi.$$

It follows that $\langle \varphi, a \rangle \langle D(b), a \rangle = \langle \varphi, b \rangle \langle D(a), a \rangle + \langle \varphi, a \rangle \langle D(b), a \rangle$, from which we have $\langle D(a), a \rangle = 0$, or equivalently, $\langle D(a + b), a + b \rangle = 0$. Therefore $\langle D(a), b \rangle = -\langle D(b), a \rangle$. Now with e as a left identity for ${}_{\varphi}A$ we have

$$D(b) = D(eb) = \langle \varphi, b \rangle D(e) + \langle D(b), e \rangle \varphi = \langle \varphi, b \rangle D(e) - \langle D(e), b \rangle \varphi = \delta_{-D(e)}(b).$$

Therefore D is inner, as required.

To prove that ${}_{\varphi}A$ is not $(2n)$ -weakly amenable for any $n \in \mathbb{N}$, it is enough to show that ${}_{\varphi}A$ is not 2-weakly amenable, [2, Proposition 1.2]. To this end let $f \in ({}_{\varphi}A)^*$ such that f and φ are linearly independent. It follows that $\langle f, a_0 \rangle = \langle \varphi, b_0 \rangle = 0$ and $\langle f, b_0 \rangle = \langle \varphi, a_0 \rangle = 1$, for some $a_0, b_0 \in {}_{\varphi}A$. Define $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^{**}$ by $D(a) = \langle f - \varphi, a \rangle b_0$, then D is a derivation. If there exists $m \in ({}_{\varphi}A)^{**}$ with $D(a) = am - ma$ ($a \in {}_{\varphi}A$) then $b_0 = -\langle m, \varphi \rangle b_0$ which follows that $1 = -\langle m, \varphi \rangle$. Now if $a \in \ker \varphi$ then $\langle f, a \rangle b_0 = -\langle m, \varphi \rangle a = a$. It follows that $\dim(\ker \varphi) = 1$ that is a contradiction. \square

As an immediate consequence of Theorem 2 we give the next result of Zhang.

Corollary 3 ([6, Page 507]). *For each $n \in \mathbb{N}$, ${}_{\varphi_{s_0}}l^1(S)$ is $(2n - 1)$ -weakly amenable, while it is not $(2n)$ -weakly amenable for any $n \in \mathbb{N}$.*

Proposition 4. *A bounded linear map $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^{(2n)}$, $(n \in \mathbb{N})$, is a derivation if and only if $D({}_{\varphi}A) \subseteq \ker \varphi$.*

Proof. A direct verification shows that $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^{(2n)}$ is a derivation if and only if

$$\langle \varphi, a \rangle D(b) = D(ab) = D(a)b + aD(b) = \langle D(a), \varphi \rangle b + \langle \varphi, a \rangle D(b) \quad (a, b \in {}_{\varphi}A).$$

And this is equivalent to $\langle D(a), \varphi \rangle = 0$, $(a \in {}_{\varphi}A)$; that is $D({}_{\varphi}A) \subseteq \ker \varphi$. Note that here φ is assumed to be an element of $({}_{\varphi}A)^{(2n+1)}$. \square

The next results demonstrates that in contrast to Theorem 2, ${}_{\varphi}A$ is $(2n)$ -weakly amenable in the case where $\dim(\ker \varphi) < 2$.

Proposition 5. *If $\dim(\ker \varphi) < 2$ then ${}_{\varphi}A$ is $(2n)$ -weakly amenable for each $n \in \mathbb{N}$.*

Proof. The only reasonable case that we need to verify is $\dim(\ker \varphi) = 1$. In this case we have $\dim(A) = 2$. Therefore one may assume that A is generated by two elements $e, a \in A$ such that $\langle \varphi, e \rangle = 1$ and $\langle \varphi, a \rangle = 0$. Let $f \in ({}_{\varphi}A)^*$ satisfy $\langle f, e \rangle = 0$ and $\langle f, a \rangle = 1$. Then f and φ are linearly independent and generate A^* ; indeed, every non-trivial element $g \in A^*$ has the form $g = \langle g, e \rangle \varphi + \langle g, a \rangle f$. Let $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^{(2n)}$ be a derivation then as Proposition 5 demonstrates $D({}_{\varphi}A) \subseteq \ker \varphi$. Therefore $D(x) = \langle g, x \rangle a$, $(x \in {}_{\varphi}A)$, for some $g \in ({}_{\varphi}A)^*$. As for every $x \in {}_{\varphi}A$, $x = \langle \varphi, x \rangle e + \langle f, x \rangle a$, a direct calculation reveals that $D = \delta_{(\langle g, e \rangle a - \langle g, a \rangle e)}$; as required. \square

Remark 6. (i) If we go through the proof of Theorem 2 we see that the range of the derivation $D(a) = \langle f - \varphi, a \rangle b_0$ lies in ${}_{\varphi}A$, and the same argument may apply to show that it is not inner as a derivation from ${}_{\varphi}A$ to ${}_{\varphi}A$. This shows that ${}_{\varphi}A$ is not 0-weakly amenable, i.e. $H^1({}_{\varphi}A, {}_{\varphi}A) \neq 0$; see a remark just after [2, Proposition 1.2]. The same situation has occurred in the proof of Proposition 5.

(ii) As ${}_{\varphi}A$ has a left identity and it is a left ideal in $({}_{\varphi}A)^{**}$, it is worthwhile mentioning that, to prove the $(2n - 1)$ -weak amenability of ${}_{\varphi}A$ it suffices to show that ${}_{\varphi}A$ is weakly amenable; [6, Theorem 3], and this has already done by Dales *et al.* [3, Page 713].

We have seen in the first part of the proof of Theorem 2 that if $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^{(2n-1)}$ is a derivation then $\langle D(a), b \rangle + \langle D(b), a \rangle = 0$, $(a, b \in {}_{\varphi}A)$; and the latter is known as a cyclic derivation for the case $n = 1$. Therefore as a consequence of Theorem 2 we get:

Corollary 7. *A bounded linear mapping $D : {}_{\varphi}A \rightarrow ({}_{\varphi}A)^*$ is a derivation if and only if it is a cyclic derivation. In particular, ${}_{\varphi}A$ is cyclicly amenable.*

We conclude with the following list consist of some miscellaneous properties of ${}_{\varphi}A$ which can be verified straightforwardly.

- If $\varphi = \lambda\psi$ for some $\lambda \in \mathbb{C}$ then trivially ${}_{\varphi}A$ and ${}_{\psi}A$ are isomorphic; indeed, the mapping $a \rightarrow \lambda a$ defines an isomorphism. However, the converse is not valid, in general. For instance, let A be generated by two elements a, b . Choose $\varphi, \psi \in A^*$ such that $\langle \varphi, a \rangle = \langle \psi, b \rangle = 0$ and $\langle \varphi, b \rangle = \langle \psi, a \rangle = 1$, then ${}_{\varphi}A$ and ${}_{\psi}A$ are isomorphic (indeed, $\alpha a + \beta b \rightarrow \alpha b + \beta a$ defines an isomorphism), however φ and ψ are linearly independent.
- It can be readily verified that $\{0\} \cup \{a \in {}_{\varphi}A, \varphi(a) = 1\}$ is the set of all idempotents of ${}_{\varphi}A$. Moreover, this is actually the set of all minimal idempotents of ${}_{\varphi}A$.
- It is obvious that every subspace of ${}_{\varphi}A$ is a left ideal, while a subspace I is a right ideal if and only if either $I = {}_{\varphi}A$ or $I \subseteq \ker \varphi$. In particular, $\ker \varphi$ is the unique maximal ideal in ${}_{\varphi}A$.
- A subspace I of ${}_{\varphi}A$ is a modular left ideal if and only if either $I = {}_{\varphi}A$ or $I = \ker \varphi$. In particular, $\ker \varphi$ is the unique primitive ideal in ${}_{\varphi}A$ and this implies that $\text{rad}({}_{\varphi}A) = \ker \varphi$ and so ${}_{\varphi}A$ is not semi-simple. Furthermore, for every non-zero proper closed ideal I , $\text{rad}(I) = I \cap \text{rad}({}_{\varphi}A) = I \cap \ker \varphi = I$.
- A direct verification reveals that $LM({}_{\varphi}A) = \mathbb{C}I$ and $RM({}_{\varphi}A) = B({}_{\varphi}A)$, where LM and RM stand for the left and right multipliers, respectively.

REFERENCES

- [1] A. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.
- [2] H.G. Dales, F. Ghahramani, and N. Grønbaek, *Derivations into iterated duals of Banach algebras*, Studia. Math **128** (1) (1998), 19–54.
- [3] H.G. Dales, A. Rodrigues-Palacios and M.V. Velasco, *The second transpose of a derivation*, J. London Math. Soc. **64** (2) (2001), 707–721.
- [4] N. Grønbaek, *Weak and cyclic amenability for non-commutative Banach algebras*, Proc. Edinburgh Math. Soc. **35** (1992), 315–328.

- [5] S. Mohammadzadeh and H.R.E. Vishki, *Arens regularity of module actions and the second adjoint of a derivation*, Bull. Austral. Math. Soc. **77** (2008), 465–476.
- [6] Y. Zhang, *Weak amenability of a class of Banach algebras*, Canad. Math. Bull. **44** (4) (2001), 504–508.

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